Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

Remarks on the Logarithmic Integrals of Regular Linear Differential Equations.

By Karl Heun, *Munich.*

Logarithms generally appear in the expressions for the solutions of regular linear differential equations when two or more roots of the fundamental equation become equal to each other. But if the corresponding indices differ by integers (not including zero), these logarithms may disappear, provided that certain conditions be satisfied. Fuchs has expressed these conditions in the form of determinants (Journal für Mathematik, LXVIII, pag. 376). I noticed, however, some time ago, that the Fuchs equations are not independent of each other when the proposed differential equation is of a higher order than the second. In the present paper the minimum number of conditions on which the existence of logarithms depends is deduced from very elementary principles. Besides this, several theorems concerning the pseudo-singular points (points à apparence singulière) of linear differential equations of the second order are given in such an explicit form as to facilitate practical applications to concrete cases.

1.

Any regular linear differential equation may be reduced to the form (cf. my paper "Zur Theorie der mehrwerthigen, mehrfach lineär verknüpften Functionen" in Acta Mathematica, t. XI, pag. 97),

$$[\psi(x)]^p \cdot \frac{d^p y}{dx^p} + [\psi(x)]^{p-1} \cdot F_1(x) \cdot \frac{d^{p-1} y}{dx^{p-1}} + \ldots + F_p(x) \cdot y = 0, \tag{A}$$

$F_\pi(x)$ denoting an entire function of the degree $\pi(i-1)$. The i roots of the equation

$$\psi(x) = (x-\xi_1)(x-\xi_2) \ldots (x-\xi_i) = 0,$$

together with $\xi_{i+1} = \infty$ mark the "singular points" of the integral function y which satisfies the preceding differential equation.

Let $(y_{1\mathrm{i}}, y_{2\mathrm{i}}, \ldots . y_{p\mathrm{i}})$ be a system of fundamental integrals of equation (A), representing its complete solution in the neighborhood of the singular point $\xi_\mathrm{i}\,(\mathrm{i} = 1, 2, \ldots . i, i+1)$, and suppose the point P, the geometrical representation of the argument x, to describe round the singular point ξ_i, a closed curve whose interior does not contain any other singular point $\xi_\iota\,(\iota \gtrless \mathrm{i})$, then the integrals $y_{1\mathrm{i}}, y_{2\mathrm{i}}, \ldots . y_{p\mathrm{i}}$ will not return to their initial values when the point P has completed its circuit, but will acquire the new values

$$\bar{y}_{1\mathrm{i}} = w_\mathrm{i}^{(1)}.y_{1\mathrm{i}},\ \bar{y}_{2\mathrm{i}} = w_\mathrm{i}^{(2)}.y_{2\mathrm{i}},\ \ldots .\ \bar{y}_{p\mathrm{i}} = w_\mathrm{i}^{(p)}.y_{p\mathrm{i}}. \tag{a}$$

The coefficients $w_\mathrm{i}^{(1)}, w_\mathrm{i}^{(2)}, \ldots . w_\mathrm{i}^{(p)}$ are the roots of a certain algebraic equation which is generally termed the "fundamental equation."

Whenever two or more (π) roots of this equation become equal to each other, the equations (a) will have to be replaced by the following system (cf. Fuchs, Journal für Mathemat., t. LXVI, pag. 136]:

$$\left.\begin{array}{ll}
\bar{y}_{1\mathrm{i}} & = w_\mathrm{i} \cdot y_{1\mathrm{i}}, \\
\bar{y}_{2\mathrm{i}} & = w_\mathrm{i} \cdot y_{2\mathrm{i}} + \tilde{\omega}_{11} y_{1\mathrm{i}}, \\
\bar{y}_{3\mathrm{i}} & = w_\mathrm{i} \cdot y_{3\mathrm{i}} + \tilde{\omega}_{21} y_{2\mathrm{i}} + \tilde{\omega}_{12} y_{1\mathrm{i}}, \\
\ldots\ldots & \ldots\ldots\ldots\ldots\ldots \\
\bar{y}_{\pi,\mathrm{i}} & = w_\mathrm{i} y_{\pi\mathrm{i}} + \tilde{\omega}_{\pi-1,1} \cdot y_{\pi-1,\mathrm{i}} + \tilde{\omega}_{\pi-2,2} \cdot y_{\pi-2,\mathrm{i}} + \ldots . + \tilde{\omega}_{11\pi-1} \cdot y_{1\mathrm{i}}, \\
\bar{y}_{\pi+1,\mathrm{i}} & = w_\mathrm{i}^{(\pi+1)} . y_{\pi+1,\mathrm{i}}, \\
\ldots\ldots & \ldots\ldots\ldots \\
\bar{y}_{p,\mathrm{i}} & = w_\mathrm{i}^{(p)} . y_{p,\mathrm{i}}.
\end{array}\right\} \quad \text{(I)}$$

In these formulas $\tilde{\omega}_{11}, \tilde{\omega}_{21}, \tilde{\omega}_{12}, \ldots . \tilde{\omega}_{1,\pi-1}$ are constant quantities and

$$w_\mathrm{i} = w_\mathrm{i}^{(1)} = w_\mathrm{i}^{(2)} = \ldots . = w_\mathrm{i}^{(\pi)}. \tag{b}$$

The equations (I) will be satisfied by the following expressions for $y_{1\mathrm{i}}, y_{2\mathrm{i}}, \ldots . y_{p\mathrm{i}}$:

$$\left.\begin{array}{ll}
y_{1\mathrm{i}} & = \eta_{1\mathrm{i}}, \\
y_{2\mathrm{i}} & = \eta_{2\mathrm{i}} + c_{11} \cdot \eta_{1\mathrm{i}} . \lg(x - \xi_\mathrm{i}), \\
y_{3\mathrm{i}} & = \eta_{3\mathrm{i}} + c_{21} \cdot \eta_{2\mathrm{i}} . \lg(x - \xi_\mathrm{i}) + c_{12} \cdot \eta_{1\mathrm{i}} . [\lg(x - \xi_\mathrm{i})]^2, \\
\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
y_{\pi,\mathrm{i}} & = \eta_{\pi,\mathrm{i}} + c_{\pi-1,1} \cdot \eta_{\pi-1,\mathrm{i}} . \lg(x - \xi_\mathrm{i}) + \ldots . + c_{11\pi-1} \cdot \eta_{1\mathrm{i}} . [\lg(x - \xi_\mathrm{i})]^{\pi-1}, \\
y_{\pi+1,\mathrm{i}} & = \eta_{\pi+1,\mathrm{i}}, \\
\ldots\ldots & \\
y_{p,\mathrm{i}} & = \eta_{p,\mathrm{i}}.
\end{array}\right\} \quad \text{(II)}$$

The function $\eta_{\mathfrak{p}\mathrm{i}}\,(\mathfrak{p} = 1,\ 2,\ \ldots .\ p)$ is defined by the expression

$$\eta_{\mathfrak{p}\mathrm{i}} = (x - \xi_\mathrm{i})^{\lambda_{\mathfrak{p}\mathrm{i}}} . \,\mathfrak{P}_\mathfrak{p}\,(x - \xi_\mathrm{i}), \tag{c}$$

$\mathfrak{P}_\mathfrak{p}\,(x - \xi_\mathrm{i})$ being the usual symbol for the convergent series

$$a_0^{(\mathfrak{p})} + a_1^{(\mathfrak{p})}\,(x - \xi_\mathrm{i}) + a_2^{(\mathfrak{p})}\,(x - \xi_\mathrm{i})^2 + \ldots . + \text{in inf.},$$

where $a_0^{(\mathfrak{p})}$ is always different from zero.

The indices $\lambda_{\mathfrak{p}\mathrm{i}}\,(\mathfrak{p} = 1,\ 2,\ \ldots .\ p)$ are the roots of the equation

$$\left.\begin{aligned} &[\psi'(\xi_\mathrm{i})]^p \lambda\,(\lambda - 1) \ldots .\,(\lambda - p + 1) \\ &+ \sum_{\mathfrak{p}=1}^{\mathfrak{p}=p-1} [\psi'\,(\xi_\mathrm{i})]^{p-\mathfrak{p}} . F_\pi\,(\xi_\mathrm{i}) . \lambda\,(\lambda - 1) \ldots .\,(\lambda - p + \pi + 1) + F_p(\xi_\mathrm{i}) = 0 . \end{aligned}\right\} \tag{B}$$

Because $\lambda_{\mathfrak{p}\mathrm{i}}$ and $w_\mathrm{i}^{(\mathfrak{p})}$ are connected by the relation

$$\lambda_{\mathfrak{p}\mathrm{i}} = \frac{1}{2\pi\sqrt{-1}} . \lg w_\mathrm{i}^{(\mathfrak{p})}, \tag{d}$$

the indices $\lambda_{1,\,\mathrm{i}}, \lambda_{2,\,\mathrm{i}}, \ldots . \lambda_{\pi,\,\mathrm{i}}$ can only differ from one another by integers.

The constant coefficients $c_{11}, c_{21}, c_{12}, \ldots .\ c_{1,\,\pi-1}$ in equations (II) are subjected to certain conditions, unless $\pi = 2$. In fact we derive from the system (II)

$$\bar{y}_{3\mathrm{i}} = \bar{\eta}_{3\mathrm{i}} + c_{21}\bar{\eta}_{2\mathrm{i}} . \left[\lg\,(x - \xi_\mathrm{i}) + 2\pi\sqrt{-1}\right] + c_{12}\bar{\eta}_{1\mathrm{i}}\left[\lg\,(x - \xi_\mathrm{i}) + 2\pi\sqrt{-1}\right]^2 .$$

But from equation (c) we conclude

$$\bar{\eta}_{\mathfrak{p}\mathrm{i}} = w_i . \eta_{\mathfrak{p}\mathrm{i}} \,|\, \mathfrak{p} = 1,\ 2,\ \ldots .\ \pi .$$

Hence

$$\begin{aligned} \bar{y}_{3\mathrm{i}} &= w_\mathrm{i}\eta_{3\mathrm{i}} + c_{21}w_\mathrm{i} . \eta_{2\mathrm{i}}\left[\lg\,(x - \xi_\mathrm{i}) + 2\pi\sqrt{-1}\right] \\ &\quad + c_{12}w_\mathrm{i} . \eta_{1\mathrm{i}}\left[\lg\,(x - \xi_\mathrm{i}) + 2\pi\sqrt{-1}\right]^2 \\ &= w_\mathrm{i}\eta_{3\mathrm{i}} + 2\pi\sqrt{-1}\,w_\mathrm{i}\left\{c_{21}\eta_{2\mathrm{i}} + 2c_{12}\eta_{1\mathrm{i}} . \lg\,(x - \xi_\mathrm{i})\right\} \\ &\quad + (2\pi\sqrt{-1})^2 w_i . \eta_{1i}\left[\lg\,(x - \xi_i)\right]^2 . \end{aligned}$$

The last expression for $\bar{y}_{3\mathrm{i}}$ will only coincide with

$$w_\mathrm{i} . \eta_{3\mathrm{i}} + \tilde{\omega}_{21}\eta_{2\mathrm{i}} + \tilde{\omega}_{12}\eta_{1i}$$

if the condition $2c_{12} = c_{21} . c_{11}$ be fulfilled. By continuing this simple process we get the remaining relations

$$2c_{22} = c_{31} . c_{21},\quad 3c_{13} = c_{22} . c_{11},\ \ldots .\ \pi c_{1,\,\pi-1} = c_{2,\,\pi-2} . c_{11} .$$

The number of the coefficients $c_{11}, c_{21}, c_{12}, \ldots .\ c_{1,\,\pi-1}$ is $\frac{1}{2}\,\pi\,(\pi - 1)$. As there are $\frac{1}{2}(\pi - 1)(\pi - 2)$ conditions between them, only $\pi - 1$ of them are independent of one another.

Now it is evident that y_{2i} will contain no logarithm if $c_{11} = 0$. Likewise y_{3i} will be free of logarithms if $c_{11} = c_{21} = 0$. Generally the integral of the form

$$y_{\pi i} = \eta_{\pi i} + c_{\pi-1,1} \cdot \eta_{\pi-1,i} \cdot \lg(x - \xi_i) + \ldots\ldots + c_{1,\pi-1} \cdot \eta_{1i} \cdot [\lg(x - \xi_i)]^{\pi-1}$$

can be reduced to the form $y_{\pi i} = \eta_{\pi i}$ if the preceding integrals $y_{\pi-1,i}, \ldots\ldots y_2$ contain no logarithms and if the additional condition $c_{\pi-1,1} = 0$ be fulfilled.

By combining this result with a well known theorem of Fuchs (Journal für Mathematik, t. LXVIII, pag. 367), we obtain the following theorem:

I. "If the indices $\lambda_{1i}, \lambda_{2i}, \ldots\ldots \lambda_{\pi i}$ belonging to the functions $y_{1i}, y_{2i}, \ldots\ldots y_{\pi}$ differ from one another by integer numbers, the analytical expressions of those integrals will, as a rule, contain logarithms (y_{1i} excepted). But unless two of the indices $\lambda_{1i}, \lambda_{2i}, \ldots\ldots \lambda_{\pi i}$ are equal to one another, they may be freed from the logarithms by satisfying certain conditions. The number of these conditions is $\pi - 1$."

We have not yet proved that the expressions given in equations (II) for $y_{1i}, y_{2i}, \ldots\ldots y_{pi}$ form really a complete system of fundamental integrals of the differential equation (A). For that purpose, let us consider the following expressions:

$$y_{1i} \cdot (x - \xi_i)^{-\lambda_{1i}},\ y_{2i} \cdot (x - \xi_i)^{-\lambda_{2i}},\ \ldots\ldots\ y_{\pi i} \cdot (x - \xi_i)^{-\lambda_{\pi i}}.$$

If $\lambda_{1i} > \lambda_{2i} > \ldots\ldots > \lambda_{\pi i}$, we may write

$$\lambda_{\pi-1,i} - \lambda_{\pi i} = n_1,\ \lambda_{\pi-2,i} - \lambda_{\pi i} = n_2,\ \ldots\ldots\ \lambda_{1i} - \lambda_{\pi i} = n_{\pi-1},$$

where $n_1, n_2, \ldots\ldots n_{\pi-1}$ are certain positive integers subject to the conditions

$$n_1 < w_2,\ n_2 < n_3,\ \ldots\ldots\ n_{\pi-2} < n_{\pi-1},$$

or

$$n_1 \geqq 1,\quad n_2 \geqq 2,\ \ldots\ldots\ n_{\pi-1} \geqq \pi - 1.$$

For any index $\tilde{\omega} \leqq \pi$ we have by equation (c),

$$\begin{aligned} y_{\hat{\omega} i} = (x - \xi_i)^{\lambda_{\hat{\omega} i}} \{ \mathfrak{P}_{\hat{\omega}}(x - \xi_i) &+ c_{\hat{\omega}-1,1} \cdot (x - \xi_i)^{n_1} \cdot \mathfrak{P}_{\hat{\omega}-1}(x - \xi_i) \cdot \lg(x - \xi_i) \\ &+ c_{\hat{\omega}-2,2} \cdot (x - \xi_i)^{n_2} \cdot \mathfrak{P}_{\hat{\omega}-2}(x - \xi_i) \cdot [\lg(x - \xi_i)]^2 \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ c_{1,\hat{\omega}-1} \cdot (x - \xi_i)^{n_{\hat{\omega}-1}} \cdot \mathfrak{P}_1(x - \xi_i) \cdot [\lg(x - \xi_i)]^{\hat{\omega}-1} \}. \end{aligned}$$

Hence we conclude that the expression $y_{\hat{\omega} i} \cdot (x - \xi_i)^{-\lambda_{\hat{\omega} i}} = \Omega_{\hat{\omega}}$ will be a finite continuous function in the neighborhood of the singular point ξ_i, for we have under the preceding conditions,

$$[(x-\xi_i)^{n_y}.[\lg(x-\xi_i)]^{y-1}]_{x=\xi_i}=0,$$
$$(y=1,\ 2,\ \ldots.\ \tilde{\omega}).$$

From the equation

$$y_{\hat{\omega}i}=(x-\xi_i)^{\lambda_{\hat{\omega}i}}.\Omega_{\hat{\omega}}$$

we derive by differentiation

$$\frac{dy_{\hat{\omega}i}}{dx}=(x-\xi_i)^{\lambda_{\hat{\omega}i}-1}.\left\{\lambda_{\hat{\omega}i}.\Omega_{\hat{\omega}}+(x-\xi_i).\frac{d\Omega_{\hat{\omega}}}{dx}\right\}.$$

It will be easily seen that the expression enclosed in brackets has the same characteristic properties as the function $\Omega_{\hat{\omega}}$ itself. On this account we may write

$$\frac{dy_{\hat{\omega}i}}{dx}=(x-\xi_i)^{\lambda_{\hat{\omega}i}-1}.\Omega_{\hat{\omega}}^{(\prime)},$$

and likewise

$$\frac{d^2y_{\hat{\omega}i}}{dx^2}=(x-\xi_i)^{\lambda_{\hat{\omega}i}-2}.\Omega_{\hat{\omega}}^{(\prime\prime)},\ \ldots.\ \frac{d^py_{\hat{\omega}i}}{dx^p}=(x-\xi_i)\,.^{\lambda_{\hat{\omega}i}-p}.\Omega_{\hat{\omega}}^{(p)},$$

$\Omega_{\hat{\omega}}^{(\prime)}.\Omega_{\hat{\omega}}^{(\prime\prime)},\ \ldots.\ \Omega_{\hat{\omega}}^{(p)}$ denoting a series of functions analogous to $\Omega_{\hat{\omega}}$. Hence we prove in the usual manner that y_{1i}, y_{2i}, ... $y_{\pi i}$, together with $y_{\pi+1,i}$, $y_{\pi+2,i}$, ... y_{pi}, satisfy a regular linear differential equation of the order p (cf. Fuchs' paper in Vol. 66 of the Journal für Mathematik, pag. 142–144).

The case $\pi=p$ requires our special attention. If now y_{2i}, y_{3i}, ... y_{pi} do not involve any logarithms, the point ξ_i will cease to be a singular point for the ratios $y_{1i}:y_{pi}$, $y_{2i}:y_{pi}$, ... $y_{p-1,i}:y_{pi}$. It may be called a "pseudo-singular point" of the differential equation (A). Remembering that the indices λ_{1i}, λ_{2i}, ... λ_{pi} are now such as to satisfy the $p-1$ conditions

$$\lambda_{p-1,i}-\lambda_{pi}=n_1,\ \lambda_{p-2,i}-\lambda_{pi}=n_2,\ \ldots.\ \lambda_{1i}-\lambda_{pi}=n_{p-1}; \qquad \text{(C)}$$

n_1, n_2, ... n_{p-1} being positive integers, we conclude from theorem I:

II. "In order that the point ξ_i be a *pseudo-singular point* of the linear equation (A), it is necessary and sufficient to satisfy $2p-2$ conditions."

We shall have now to establish $p-1$ of these conditions "explicitly" so as to acquire a sufficient criterion, whether a certain point ξ_i be a pseudo-singular point of a regular differential equation or not, supposing that the $p-1$ conditions (C) are fulfilled *a priori*. This problem will be solved gradually by beginning with equations of the second order.

2.

In the particular case $p=2$ equation (A) takes the simple form

$$[\psi(x)]^2.\frac{d^2y}{dx^2}+\psi(x).F_1(x).\frac{dy}{dx}+F_2(x).y=0. \tag{1}$$

Equation (B) becomes

$$[\psi'(\xi_{\mathrm{i}})]^2\lambda(\lambda-1)+\psi'(\xi_{\mathrm{i}}).F_1(\xi_{\mathrm{i}}).\lambda+F_2(\xi_{\mathrm{i}})=0, \tag{2}$$
$$\mathrm{i}=1,\ 2,\ \ldots .\ i.$$

The analogous equation belonging to the point $\xi_{i+1}=\infty$ is

$$\lambda(\lambda+1)-\left[\frac{F_1(x)}{x^{i-1}}\right]_{x=\infty}.\lambda+\left[\frac{F_2(x)}{x^{2i-2}}\right]_{x=\infty}=0. \tag{3}$$

From this equation (whose roots are $\lambda_{1,i+1}$, $\lambda_{2,i+1}$) follows

$$\left[\frac{F_2(x)}{x^{2i-2}}\right]_{x=\infty}=\lambda_{1,i+1}.\lambda_{2,i+1},$$

whilst we get from equation (2)

$$F_2(\xi_{\mathrm{i}})=\lambda_{1,\mathrm{i}}.\lambda_{2,\mathrm{i}}\,|\,\mathrm{i}=1,\ 2,\ \ldots .\ i.$$

The integer function $F_2(x)$ will therefore be divisible by $\psi(x)$ if

$$\lambda_{11}=\lambda_{12}=\ldots .=\lambda_{1,i-1}=\lambda_{1i}=0.$$

But it is very easy to express the general integral of equation (1) by another function whose indices satisfy the latter condition. Indeed, denoting every general solution of a regular linear differential equation by the tables of its indices in this manner

$$\left\{\begin{matrix}\lambda_{11}, & \lambda_{12}, & \ldots . & \lambda_{1i}, & \lambda_{1,i+1}\\ \lambda_{21}, & \lambda_{22}, & \ldots . & \lambda_{2i}, & \lambda_{2,i+1}\end{matrix}\right\}$$

we may verify immediately the following relation:

$$\left\{\begin{matrix}\lambda_{11}, & \lambda_{12}, & \ldots . & \lambda_{1i}, & \lambda_{1,i+1}\\ \lambda_{21}, & \lambda_{22}, & \ldots . & \lambda_{2i}, & \lambda_{2,i+1}\end{matrix}\right\}=(x-\xi_1)^{\lambda_{11}}.(x-\xi_2)^{\lambda_{12}}.\ldots .(x-\xi_i)^{\lambda_{1i}}$$
$$\times\left\{\begin{matrix}0 & ,\ \ldots . & 0 & \lambda_{1,i+1}+\sum\limits_{\mathrm{i}=1}^{\mathrm{i}=i}\lambda_{1\mathrm{i}}\\ \lambda_{21}-\lambda_{11}, & \ldots . & \lambda_{2i}-\lambda_{1i}, & \lambda_{2,i+1}+\sum\limits_{\mathrm{i}=1}^{\mathrm{i}=i}\lambda_{1\mathrm{i}}\end{matrix}\right\}$$

If we put therefore

$$y=(x-\xi_1)^{\lambda_{11}}.(x-\xi_2)^{\lambda_{12}}.\ldots.(x-\xi_i)^{\lambda_{1i}}.\eta\,,$$

the function η will satisfy the differential equation

$$\psi(x)\,.\,\frac{d^2\eta}{dx^2}+\chi(x)\,.\,\frac{d\eta}{dx}+\tilde{\omega}(x)\,.\,\eta=0\,, \tag{1a}$$

$$\left[\chi(x)=F_1(x)\,;\;\;\tilde{\omega}(x)=\frac{F_2(x)}{\psi(x)}\right],$$

$\tilde{\omega}(x)$ is an integer function of the degree $i-2$. On account of equation (3a) it has the form

$$\tilde{\omega}(x)=\mathfrak{l}_{1,\,i+1}\,.\,\mathfrak{l}_{2,\,i+1}\,.\,x^{i-2}+\varkappa_1x^{i-3}+\varkappa_2x^{i-4}+\ldots.+\varkappa_{i-3}x+\varkappa_{i-2} \tag{1b}$$

if the function η is denoted by

$$\left\{\begin{matrix}0, & 0, & \ldots. & 0, & \mathfrak{l}_{1,\,i+1}\\ \mathfrak{l}_1, & \mathfrak{l}_2, & \ldots. & \mathfrak{l}_i, & \mathfrak{l}_{2,\,i+1}\end{matrix}\right\}.$$

The function $\chi(x)$ is determined by the equations

$$\chi(\xi_\mathrm{i})=-\psi'(\xi_\mathrm{i})\,\mathfrak{l}_\mathrm{i}. \tag{2a}$$

The constant coefficients $\varkappa_1,\ \varkappa_2,\ \ldots.\ \varkappa_{i-2}$ are independent of the indices $\mathfrak{l}$. According to the special values which are attributed to them, the character of the function η will be more or less complicated. For this reason we shall term them the *characteristic parameters* of the differential equation (1a).

After this digression we return to the original differential equation (1). Let ξ_ι be any one of the points $\xi_1,\ \xi_2,\ \ldots.\ \xi_i$ and

$$y_{1\iota}=(x-\xi_\iota)^{\lambda_{1\iota}}.\,\mathfrak{P}_1(x-\xi_\iota) \tag{4}$$

be one of the fundamental integrals belonging to the point ξ_ι. Then the second integral $y_{2\iota}$ may be represented by the expression

$$y_{2\iota}=y_{1\iota}\,.\int\frac{e^{-\int\frac{F_1(x)}{\psi(x)}dx}}{y_{1\iota}\,.\,y_{1\iota}}\,dx\,. \tag{5}$$

Since the roots of the equation $\psi(x)=0$ are different from one another, we have

$$\frac{F_1(x)}{\psi(x)}=\sum_{\mathrm{i}=1}^{\mathrm{i}=i}\frac{F_1(\xi_\mathrm{i})}{\psi'(\xi_\mathrm{i})}\,.\,\frac{1}{x-\xi_\mathrm{i}}\,. \tag{6}$$

But, on account of equation (2), we find directly

$$F_1(\xi_\mathrm{i})=-\left[\lambda_{1\mathrm{i}}+\lambda_{2\mathrm{i}}-1\right].\,\psi'(\xi_\mathrm{i})\,,$$

consequently

$$e^{-\int\frac{F_1(x)}{\psi(x)}dx} = \prod_{i=1}^{i=i}(x-\xi_i)^{\lambda_{1i}+\lambda_{2i}-1}.$$

Writing for brevity

$$P(x) = \frac{\prod\limits_{i=1}^{i=i}(x-\xi_i)^{\lambda_{1i}+\lambda_{2i}-1}}{(x-\xi_\iota)^{\lambda_{1\iota}+\lambda_{2\iota}-1}} \cdot \frac{1}{\mathfrak{P}_1(x-\xi_\iota) \,.\, \mathfrak{P}_1(x-\xi_\iota)}, \tag{7}$$

equation (5) can be put in the form

$$y_{2\iota} = y_{1\iota} \cdot \int P(x) \,.\, (x-\xi_\iota)^{\lambda_{1\iota}-\lambda_{1\iota}-1}\,dx.$$

But it is possible to develop $P(x)$ into the convergent series

$$c_0 + c_1(x-\xi_\iota) + c_2(x-\xi_\iota)^2 + \ldots\ldots + \text{ in inf.}$$

where the first constant c_0 differs necessarily from zero. Therefore we have

$$y_{2\iota} = y_{1\iota} \cdot \int \Big[\sum_{\nu=0}^{\nu=\infty} c_\nu \,.\, (x-\xi_\iota)^{\lambda_{2\iota}-\lambda_{1\iota}-1+\nu}\Big]\,dx.$$

Supposing $\lambda_{1\iota}-\lambda_{2\iota}=n$ (n being a positive integer) we obtain, by integration,

$$y_{2\iota} = y_{1\iota} \cdot \Big\{-\frac{1}{n}c_0(x-\xi_\iota)^{-n} - \frac{1}{n-1}c_1(x-\xi_\iota)^{-(n-1)} - \ldots\ldots - c_{n-1}(x-\xi_\iota)^{-1}$$
$$+ c_n \,.\, \lg(x-\xi_\iota) + c_{n+1}(x-\xi_\iota) + \frac{1}{2}c_{n+2}(x-\xi_\iota) + \ldots\ldots + \text{ in inf.}\Big\}.$$

Hence we get for $y_{2\iota}$ the final expression

$$y_{2\iota} = (x+\xi_\iota)^{\lambda_{2\iota}} \,.\, \mathfrak{P}_2(x-\xi_\iota) + c_n(x-\xi_\iota)^{\lambda_{1\iota}} \,.\, \mathfrak{P}_1(x-\xi_\iota) \,.\, \lg(x-\xi_\iota). \tag{8}$$

From this formula it is evident that $y_{2\iota}$ will contain no logarithm if the equation $c_n = 0$ be fulfilled. We have now to establish the connection of this equation with the coefficients of the differential equation (1) or, what is the same thing, with the indices λ and the characteristic parameters $\varkappa$.

We may always consider $\mathfrak{P}_1(x-\xi_\iota)$ as a known function. Let it be given in the form

$$\mathfrak{P}_1(x-\xi_\iota) = a_0 + a_1(x-\xi_\iota) + a_2(x-\xi_\iota)^2 + \ldots\ldots + \text{ in inf.}$$

$\mathfrak{P}_1(x-\xi_\iota) \,.\, \mathfrak{P}_1(x-\xi_\iota)$ will be likewise a convergent series of the form

$$s - \alpha_0 + \alpha_1(x-\xi_\iota) + \alpha_2(x-\xi_\iota)^2 + \ldots\ldots + \text{ in inf.}$$

We have $\alpha_0 = a_0 a_0$, $\alpha_1 = 2a_1$ and generally

$$\begin{aligned} \alpha_{2\nu} &= 2\left[a_0 a_{2\nu} + a_1 a_{2\nu-1} + \ldots\ldots + a_{\nu-1} a_{\nu+1}\right] + a_\nu a_\nu, \\ \alpha_{2\nu+1} &= 2\left[a_0 a_{2\nu+1} + a_1 a_{2\nu} + \ldots\ldots + a_\nu a_{\nu+1}\right]. \end{aligned}$$

The coefficients of the series

$$\frac{1}{s^2} = A_0 - A_1 (x - \xi_\iota) + A_2 (x - \xi_\iota)^2 - A_3 (x - \xi_\iota)^3 + \ldots\ldots \pm \text{ in inf.}$$

are connected with those of the series s by the formulas $[a_0 = 1]$,

$$A_0 = 1\,; \; A_1 = \alpha_1\,; \; A_2 = \alpha_1^2 - \alpha_2\,; \; A_3 = \alpha_1^3 - 2\alpha_1\alpha_2 + \alpha_3, \text{ etc.}$$

The general expression for A_n as a function of $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$ is

$$A_n = \begin{vmatrix} \alpha_1, & \alpha_2, & \alpha_3, & \ldots\ldots \alpha_n \\ 1\ , & \alpha_1, & \alpha_2, & \ldots\ldots \alpha_{n-1} \\ 0\ , & 1\ , & \alpha_1, & \ldots\ldots \alpha_{n-2} \\ 0\ , & 0\ , & 1\ , & \ldots\ldots \alpha_{n-3} \\ \ldots & \ldots & \ldots & \ldots\ldots\ldots \\ 0\ , & 0\ , & 0\ , & \ldots\ldots \alpha_1 \end{vmatrix}.$$

The expression for A_n as a function of the coefficients $a_1, a_2, \ldots\ldots a_n$ is deduced from the latter one in the following manner. In the equation

$$A_n = \sum_{\nu=1}^{\nu=n} \rho_n^{(\nu)} \, [\alpha]_\nu = \sum_{\nu=1}^{\nu=n} r_n^{(\nu)} \, [\alpha]_\nu,$$

let $[\alpha]_\nu$ be the complex of all terms of the dimension ν in respect to α and $[a]_\nu$ be the corresponding expression of the terms in a. Then the numerical coefficients $\rho_n^{(\nu)}$, $r_n^{(\nu)}$ are connected by the equation $r_n^{(\nu)} = (\nu + 1) \cdot \rho_n^{(\nu)}$. By this rule we find easily

$$A_1 = 2a_1; \; A_2 = 3a_1^2 - 2a_2; \; A_3 = 4a_1^3 - 6a_1a_2 + 2a_3, \text{ etc.}$$

Next we develop the expression $\dfrac{\prod\limits_{\mathrm{i}=1}^{\mathrm{i}=i} (x - \xi_{\mathrm{i}})^{\lambda_{1\mathrm{i}} + \lambda_{2\mathrm{i}} - 1}}{(x - \xi_\iota)^{\lambda_{1\iota} + \lambda_{2\iota} - 1}}$ into the series

$$l_0 + l_1 (x - \xi_\iota) + l_2 (x + \xi_\iota)^2 + \ldots\ldots + \text{ in inf.}$$

The product of this series into the series

$$A_0 - A_1 (x - \xi_\iota) + A_2 (x - \xi_\iota)^2 - \ldots\ldots \pm \text{ in inf.}$$

is equal to

$$l_0 A_0 + (l_1 A_0 - l_0 A_1) \cdot (x - \xi_\iota) + (l_2 A_0 - l_1 A_1 + l_0 A_2) \cdot (x - \xi_\iota)^2 + \ldots\ldots$$

Hence we derive the general expression for c_n in equation (8),

$$c_n = l_n \cdot A_0 - l_{n-1} \cdot A_1 + \dots\dots + (-1)^n l_0 \cdot A_n. \tag{10}$$

This result may be enunciated in the subsequent theorem:

III. "Let ξ_ι be any of the i zeros of the integer function $\psi(x)$ in the regular differential equation

$$[\psi(x)]^2 \frac{d^2y}{dx^2} + \psi(x) \cdot F_1(x) \cdot \frac{dy}{dx} + F_2(x) \cdot y = 0$$

and $\lambda_{1\iota}$, $\lambda_{2\iota}$ the roots of the quadratic

$$[\psi'(\xi_\iota)]^2 \lambda(\lambda - 1) + \psi'(\xi_\iota) \cdot F_1(\xi_\iota) \cdot \lambda + F_2(\xi_\iota) = 0.$$

Supposing the difference $\lambda_{1\iota} - \lambda_{2\iota}$ to be equal to a positive integer n, determine a function

$$y_{1\iota} = (x - \xi_\iota)^{\lambda_{1\iota}} \cdot \{a_0 + a_1(x - \xi_\iota) + a_2(x - \xi_\iota)^2 + \dots\dots\},$$

satisfying the proposed differential equation. Thereupon express the coefficients A_0, A_1, ... A_n in the equation

$$y_{1\iota}^{-2} \cdot (x - \xi_\iota)^{2\lambda_{1\iota}} = A_0 - A_1(x - \xi_\iota) + A_2(x - \xi_\iota)^2 \pm \dots\dots$$

as functions of the coefficients a_0, a_1, ... a_n and likewise the coefficients l_0, l_1, ... l_n in the identity

$$\frac{\prod\limits_{\mathrm{i}=1}^{\mathrm{i}=\iota} (x - \xi_{\mathrm{i}})^{\lambda_{1\mathrm{i}} + \lambda_{2\mathrm{i}} - 1}}{(x - \xi_\iota)^{\lambda_{1\iota} + \lambda_{2\iota} - 1}} = l_0 + l_1(x - \xi_\iota) + \dots\dots + l_n(x - \xi_\iota)^n + \dots\dots$$

as functions of the indices

$$\lambda_{11},\ \lambda_{12},\ \dots\dots\ \lambda_{1i},$$
$$\lambda_{21},\ \lambda_{22},\ \dots\dots\ \lambda_{2i}.$$

If the aggregate

$$l_n \cdot A_0 - l_{n-1} A_1 + \dots\dots + (-1)^n l_0 A_n$$

happens to be equal to zero, the point ξ_ι will be a pseudo-singular one of the given differential equation, and $y_{1\iota}$, $y_{2\iota}$ will consequently not constitute a system of fundamental integrals."

The present criterion is formally confined to finite values of ξ_ι. But the case $\xi_\iota = \xi_{i+1} = \infty$ may be easily reduced to the case $\xi_\iota = 0$ by changing the independent variable x in the differential equation. Indeed, if we put $x = \frac{1}{t}$ the point $t = 0$ will correspond to the point $x = \infty$.

3.

The coefficients of the differential equation (1) [No. 2] are completely determined by the indices $\lambda_{\mathfrak{p}\mathfrak{i}}$ ($\mathfrak{p} = 1, 2$; $\mathfrak{i} = 1, 2, \ldots . i, i+1$) if i be equal to 2. In this case we may put $\xi_1 = 0$ and $\xi_1 = 1$ and obtain the differential equation

$$x^2(x-1)^2\frac{d^2y}{dx^2} + x(x-1)F_1(x) \, . \, \frac{dy}{dx} + F_2(x) \, . \, y = 0. \qquad (1)$$

Introducing the function η instead of y, by the substitution

$$y = x^{\lambda_{11}}(x-1)^{\lambda_{12}} \, . \, \eta,$$

η will be an integral of the equation

$$x(x-1)\frac{d^2\eta}{dx^2} + \chi(x) \, . \, \frac{d\eta}{dx} + p \, . \, \eta = 0, \qquad (2)$$

where p denotes a constant quantity. Suppose

$$\left\{\begin{matrix} 0, & 0, & \alpha \\ 1-\gamma, & \gamma-\alpha-\beta, & \beta \end{matrix}\right\}$$

to be the table of the indices of η, $\chi(x)$ will be a linear function of x satisfying the conditions

$$\chi(0) = -\gamma; \; \chi(1) = -\gamma + \alpha + \beta + 1.$$

Therefore $\quad \chi(x) = (\alpha + \beta + 1)x - \gamma.$

From equation (1b) [No. 2] we conclude $p = \alpha\beta$. Consequently η will be a solution of the equation

$$x(x-1)\frac{d^2\eta}{dx^2} + [(\alpha + \beta + 1)x - \gamma]\frac{d\eta}{dx} + \alpha\beta \, . \, \eta = 0. \qquad (3a)$$

The complete integral-system of this well-known differential equation is

$$y_{11} = F[\alpha, \beta, \gamma, x],$$
$$y_{12} = F[\alpha, \beta, \alpha + \beta - \gamma + 1, (1-x)],$$
$$y_{13} = x^{-\alpha}F\left[\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \frac{1}{x}\right],$$
$$y_{21} = x^{1-\gamma} \, . \, F[\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x],$$
$$y_{22} = (1-x)^{\gamma-\alpha-\beta}F[\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, (1-x)],$$
$$y_{23} = x^{-\beta}F\left[\beta, \beta - \gamma + 1, \beta - \alpha + 1, \frac{1}{x}\right],$$

$F[\alpha, \beta, \gamma, x]$ denoting the hypergeometrical series of Gauss. This series has no definite value as soon as γ becomes a negative integer. Thus

$$F[\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, x] \text{ has to be rejected if } \gamma - 1 = n;$$

likewise $F[\gamma - \alpha, \gamma - \beta, \gamma - \alpha - \beta + 1, (1 - x)]$ if $\alpha + \beta - \gamma = n$,

and $F\left[\beta, \beta - \gamma + 1, \beta - \alpha + 1, \frac{1}{x}\right]$ if $\alpha - \beta = n$,

n denoting always a positive integer. In these cases we have the following expression for y_{21}, y_{22}, and y_{23}:

$$\begin{aligned}
y_{21} &= \mathfrak{P}_1(x) + c_n^{(1)}.F[\alpha, \beta, \gamma, x].\lg x && (\gamma - 1 = n),\\
y_{22} &= \mathfrak{P}_2(1-x) + c_n^{(2)}.F[\alpha, \beta, \alpha + \beta - \gamma + 1, (1-x)].\lg(1-x) && (\alpha + \beta - \gamma = n),\\
y_{23} &= \mathfrak{P}_3\left(\frac{1}{x}\right) + c_n^{(3)}.F\left[\alpha, \alpha - \gamma + 1, \alpha - \beta + 1, \frac{1}{x}\right].x^{-\alpha}.\lg\frac{1}{x} && (\alpha - \beta = n).
\end{aligned}$$

The condition $c_n^{(1)} = 0$ denotes that $x = 0$ is a pseudo-singular point of the differential equation (3a). Similarly $c_n^{(2)} = 0$, $c_n^{(3)} = 0$ will respectively indicate that $x = 1$ or $x = \infty$ are pseudo-singular points of the hypergeometrical differential equation.

We shall pursue only the case $c_n^{(1)} = 0$, as the two remaining may be absolved by following exactly the same method. The point $x = 0$ will be a pseudo-singular one if the condition

$$c_n - l_n.A_0 - l_{n-1}.A_1 + l_{n-2}.A_2 - \dots + (-1)^n l_0.A_n = 0$$

is ratified. As the coefficients $l_0, l_1, \dots l_n$ are in the present case given by the equation

$$(1-x)^{n-\alpha-\beta} = l_0 + l_1 x + l_2 x^2 + \dots + l_n x^n + \dots,$$

we find

$$l_0 = 1;\ l_1 = -(n - \alpha - \beta);\ l_2 = \frac{(n-\alpha-\beta)(n-\alpha-\beta-1)}{1.2}, \text{ etc.}$$

Furthermore, since

$$F[\alpha, \beta, n+1, x] = 1 + \frac{\alpha.\beta}{n+1}.x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.(n+1)(n+2)}x^2 + \dots,$$

we get

$$A_0 = 1,\ A_1 = \frac{2\alpha\beta}{n+1},\ A_2 = \frac{\alpha^2\beta^2}{(n+1)^2} - \frac{\alpha(\alpha+1)\beta(\beta+1)}{(n+1)(n+2)}, \dots.$$

Introducing these expressions for l_0, l_1, ... l_n and A_0, A_1, ... A_n into the equation $c_n = 0$, we obtain the latter in the form

$$c_n = (\alpha - 1)(\alpha - 2) \ldots\ldots (\alpha - n)(\beta - 1)(\beta - 2) \ldots\ldots (\beta - n) = 0\,.$$

Hence we conclude

$\alpha = 1, 2, 3, \ldots\ldots n$ and β arbitrary,
or $\beta = 1, 2, 3, \ldots\ldots n$ and α arbitrary.

This result justifies the theorem:

IV. "If the third element γ of the hypergeometrical series $F[\alpha, \beta, \gamma, x]$ which satisfies the differential equation

$$x(x-1)\frac{d^2\eta}{dx^2} + [(\alpha + \beta + 1)x - \gamma]\frac{d\eta}{dx} + \alpha\beta . \eta = 0$$

has the form $\gamma = n + 1$ (n being a positive integer), the expression

$$\mathfrak{P}(x) + F[\alpha, \beta, \gamma, x] . \lg x$$

together with $F[\alpha, \beta, \gamma, x]$ will constitute a complete solution of this differential equation in general. But supposing $n \geqq 1$ and α to be equal to one of the numbers $1, 2, \ldots\ldots n$, and β remaining an arbitrary quantity, or *vice versa*, the point $x = 0$ will cease to be a singular point of the proposed differential equation."

The "relationes inter functiones contiguas" [Gauss' Werke, t. III, pag. 130] may serve to express the series $F[\alpha, \nu, n+1, x]$ rationally by $F[\alpha, 1, 2, x]$ and $F[\alpha, 1, 3, x]$. By means of the formulas

$$\begin{aligned}
&(\nu - 1)(1 - x)F[\alpha, \nu, n + 1, x]\\
&+ [n - 2\nu - 1 - (\alpha - \nu + 1)x] . F[\alpha, \nu - 1, n + 1, x]\\
&- (n - \nu + 2)F[\alpha, \nu - 2, n + 1, x] = 0 \qquad \text{(form. (1) l. c.)},\\
&(n - \alpha) . x . F[\alpha, \nu, n + 1, x]\\
&+ n[n - 1 - (2n - \alpha - \nu - 1)x] . F[\alpha, \nu, n, x]\\
&- n(n - 1)(1 - x) . F[\alpha, \nu, n - 1, x] \qquad \text{(form. (15) l. c.)}
\end{aligned}$$

$F[\alpha, \nu, n+1, x]$ is expressible by

$$F[\alpha, 1, 2, x],\ F[\alpha, 1, 3, x],\ F[\alpha, 2, 2, x],\ F[\alpha, 2, 3, x]\,.$$

But between these four functions there exist two relations:

$$[1-(2-\alpha)x]\,.\,F[\alpha,\,2,\,3,\,x]+F[\alpha,\,1,\,3,\,x]-2(1-x)\,.\,F[\alpha,\,2,\,2,\,x]=0$$

(cf. form. (14) l. c.),

$$2\,[1-(2-\alpha)\,x]\,.\,F[\alpha,\,1,\,2,\,x]-2(1-x)\,.\,F[\alpha,\,2,\,2,\,x] \\ +(2-\alpha)x\,.\,F[\alpha,\,1,\,3,\,x]=0$$

(cf. form. (11) l. c.).

Therefore $F[\alpha,\,\nu,\,n+1,\,x]$ can be reduced to $F[\alpha,\,1,\,2,\,x]$ and $F[\alpha,\,1,\,3,\,x]$. We find easily

$$F[\alpha,\,1,\,2,\,x]=\frac{1}{1-\alpha}\cdot\frac{1}{x}\cdot[1-(1-x)^{1-\alpha}],$$

$$F[\alpha,\,1,\,3,\,x]=\frac{2}{1-\alpha}\cdot\frac{1}{x}-\frac{2}{(2-\alpha)(1-\alpha)}\cdot\frac{1}{x^2}\,[1-(1-x)^{2-\alpha}].$$

This is sufficient to show that $F[\alpha,\,\nu,\,n+1,\,x]$ is "a rational function of x and $(1-x)^{\alpha}$" $[1\leqq\nu\leqq n]$.

4.

The general theory of differential equations of the second order with more than two finite singular points is very little elaborated up to the present. Only a few special cases have been treated by Green and Lamé, whose researches have been completed and enlarged by Liouville, Heine, Hermite, Fuchs, and other mathematicians.

Let us first consider differential equations with a single characteristic parameter. This will occur for $i=3$. Equation (1) [No. 2] assumes in the present case ($\xi_1=0$, $\xi_2=1$, $\xi_3=a$) the form

$$x^2(x-1)^2(x-a)^2\frac{d^2y}{dx^2}+x(x-1)(x-a)\,.\,F_1(x)\frac{dy}{dx}+F_2(x)\,.\,y=0. \qquad (1)$$

This equation is satisfied by a function y with the indices

$$\begin{array}{cccc} 0\,. & 2\,. & a\,. & \infty\,. \\ \hline \lambda_{11}, & \lambda_{12}, & \lambda_{13}, & \lambda_{14}, \\ \lambda_{21}, & \lambda_{22}, & \lambda_{23}, & \lambda_{24}. \end{array}$$

If we put

$$y=x^{\lambda_{11}}(x-1)^{\lambda_{12}}(x-a)^{\lambda_{13}}\,.\,\eta,$$

the new function η will be an integral of the equation

$$x(x-1)(x-a)\frac{d^2\eta}{dx^2}+\chi(x)\,.\,\frac{d\eta}{dx}+\tilde{\omega}(x)\,.\,\eta=0. \qquad (2)$$

Suppose the table of the indices of η to be

$$\left\{\begin{array}{cccc} \underline{0\,.} & \underline{1\,.} & \underline{a\,.} & \underline{\infty\,.} \\ 0, & 0, & 0, & \alpha \\ 1-\gamma, & 1-\delta, & \gamma-\alpha+\delta-\beta, & \beta \end{array}\right\}$$

By equation (2a) [No. 2] we have therefore

$$\chi(0)=\psi'(0)\,.\,\gamma,\ \chi(1)=\psi'(1)\,.\,\delta,\ \chi(a)=-\psi'(a)\,[\gamma-\alpha+\delta-\beta-1],$$

or since $\psi(x)=x(x-1)(x-a)$, these equations are

$$\chi(0)=a\gamma,\ \chi(1)=(1-a)\,\delta,\ \chi(a)=-a(a-1)[\gamma-\alpha+\delta-\beta-1],$$

consequently

$$\chi(x)=(\alpha+\beta+1)\,x^2-\{\alpha+\beta-\delta+1+a(\gamma+\delta)\}\,.\,x+a\gamma.$$

Equation (1b) [No. 2] gives for $i=3$,

$$\tilde{\omega}(x)=\alpha\beta\,.\,x+\varkappa_1.$$

Writing $-\alpha\beta g$ instead of $\varkappa_1$, the differential equation (2) will take the final form

$$\left.\begin{array}{l} x(x-1)(x-a)\,\dfrac{d^2\eta}{dx^2} \\ +\ [(\alpha+\beta+1)\,x^2-\{\alpha+\beta-\delta+1+a(\gamma+\delta)\}\,x+a\gamma]\,\dfrac{d\eta}{dx} \\ +\ \alpha\beta\,(x-g)\,\eta=0. \end{array}\right\} \tag{2a}$$

The quantity g represents the characteristic parameter of this equation. Mod. (a) may be supposed to be >1. Should it be <1, then it will be easy to transform equation (2a) into a similar one for which the first hypothesis holds good. Now let us determine the coefficients a_1, a_2 . . . . in the series

$$\eta=\sum_{\nu=0}^{\nu=\infty} a_\nu\,.\,x^\nu, \tag{3}$$

that η be a particular solution of the equation (2a). The latter equation will be identically satisfied by the series (3) if

$$\begin{aligned}\sum_{\nu=0}^{\nu=\infty} a_\nu\,[&\{\nu(\nu-1)+(\alpha+\beta+1)\,\nu+\alpha\beta\}\,x^{\nu+1} \\ &-\{\nu(\nu-1)(a+1)+\nu\,[\alpha+\beta-\delta+1+a(\gamma+\delta)]+\alpha\beta g\}\,x^\nu \\ &+\{\nu(\nu-1)\,a+\nu a\gamma\}\,x^{\nu-1}]=0.\end{aligned}$$

From this identity we derive formula

$$\left.\begin{array}{l}(\nu+\alpha-1)(\nu+\beta-1)a_{\nu-1}-\{\nu[\nu+\alpha+\beta-\delta+a(\nu+\gamma+\delta-1)]+\alpha\beta g\}\,a_\nu \\ \qquad\qquad +\,a(\nu+1)(\nu+\gamma)a_{\nu+1}=0.\end{array}\right\}(4)$$

a_1 and a_0 are connected by the equation $a\gamma a_1-\alpha\beta g a_0=0$. Equation (4) gives

$$a_\nu=\frac{1}{1.2\,....\,\nu}\cdot\frac{\alpha.\beta}{\gamma(\gamma+1)\,....\,(\gamma+\nu-1)}\cdot\frac{G^{(\nu)}_{(g)}}{a^\nu} \qquad (5)$$

where $G^{(\nu)}_{(g)}$ denotes a certain integer function of g of the degree ν. If we put

$$\left.\begin{array}{l}F[a(\alpha,\beta,\gamma,\delta)g,x] \\ \quad =1+\dfrac{\alpha\beta}{\gamma}\left\{G^{(1)}_{(g)}.\dfrac{x}{a}+\dfrac{G^{(2)}_{(g)}}{1.2.\gamma(\gamma+1)}\cdot\dfrac{x^2}{a^2}+\dfrac{G^{(3)}_{(g)}}{1.2.3.\gamma(\gamma+1)(\gamma+2)}\cdot\dfrac{x^3}{a^3}\right. \\ \qquad\quad \left.+\dfrac{G^{(4)}_{(g)}}{1.2.3.4.\gamma(\gamma+1)(\gamma+2)(\gamma+3)}\cdot\dfrac{x^4}{a^4}+\,....\,+\text{in inf.}\right.\end{array}\right\}(6)$$

$F[a(\alpha,\beta,\gamma,\delta)g,x]$ will be certainly an integral function of the differential equation (2a) if Mod. $(x)<1$. Substituting a_ν from equation (5) into equation (4) we obtain for the successive determination of the functions $G^{(3)}_{(g)}$, $G^{(4)}_{(g)}$, ..., the formula

$$\left.\begin{array}{l}G^{(\nu+1)}_{(g)}=\{\nu[\nu+\alpha+\beta-\delta+a(\nu+\gamma+\delta)]+\alpha\beta g\}\,G^{(\nu)}_{(g)} \\ \qquad\quad -a\nu(\alpha+\nu-1)(\beta+\nu-1)(\gamma+\nu-1).\,G^{(\nu-1)}_{(g)}=0,\end{array}\right\}(7)$$

starting with the functions

$$G^{(1)}_{(g)}=g,\quad G^{(2)}_{(g)}=\alpha\beta g^2+\{\alpha+\beta-\delta+1+a(\gamma+\delta)\}g-a\gamma.$$

In the special case $a=1$, $g=1$, we have

$$G^{(\nu)}_{(1)}=(\alpha+1)(\alpha+2)\,....\,(\alpha+\nu-1).(\beta+1)(\beta+2)\,....\,(\beta+\nu-1),$$

consequently

$$F[1(\alpha,\beta,\gamma,\delta)1,x]=F[\alpha,\beta,\gamma,x].$$

This relation is important, as it shows that Gauss' hypergeometrical series may be looked at as a "degeneration" of our general series $F[a(\alpha,\beta,\gamma,\delta)g,x]$.

Since the functions of Green* and Lamé† are solutions of the differential

---

[*] "On the determination of the exterior and interior attractions," etc., in the Transactions of the Cambridge Phil. Soc., 1835 (read 6th May, 1883). This paper contains already the fundamental properties of the functions which have been afterwards called by Heine "höhere Lamésche Functionen."

[†] Liouville's Journal, t. IV (1839) and "Leçons sur les fonctions inverses," Paris, 1857.

equation

$$x(x-1)(x-a)\frac{d^2w}{dx^2}+\frac{1}{2}\left[3x^2-2(a+1)x+a\right]\frac{dw}{dx}-\frac{1}{4}n(n+1)(x+g)w=0.$$

n being a positive integer, we have the expression

$$w=F\left[a\left(-\frac{1}{2}n,\ \frac{1}{2}(n+1),\ \frac{1}{2},\ \frac{1}{2}\right)g,\ x\right].$$

A second solution of the differential equation (2a) must necessarily have the form

$$\eta_{21}=x^{1-\gamma}.\mathfrak{P}(x).$$

Now an elementary transformation of the indices gives the relation*

$$\eta\begin{Bmatrix}0, & 0, & 0, & a\\ 1-\gamma, & 1-\delta, & \gamma-\alpha+\delta-\beta, & \beta\end{Bmatrix}$$

$$=x^{1-\gamma}.\eta\begin{Bmatrix}\gamma-1, & 0, & 0, & \alpha-\gamma+1\\ 0, & 1-\delta, & \gamma-\alpha+\delta-\beta, & \beta-\gamma+1\end{Bmatrix}$$

$$=x^{1-\gamma}.\eta\begin{Bmatrix}0, & 0, & 0, & \alpha-\gamma+1\\ 1-(2-\gamma), & 1-\delta, & \gamma-\alpha+\delta-\beta, & \beta-\gamma+1\end{Bmatrix}$$

Hence follows immediately

$$y_{21}=x^{1-\gamma}.F[a(\alpha-\gamma+1,\ \beta-\gamma+1,\ 2-\gamma,\ \delta)g,\ x]. \tag{8}$$

But if $\gamma=n+1$ (n being a positive integer), this second solution has to be rejected and substituted by

$$y_{21}=\mathfrak{P}_2(x)+c_n.F[a(\alpha,\ \beta,\ \gamma,\ \delta)g,\ x].\lg x.$$

The point $x=0$ will be a pseudo-singular one if the condition $c_n=0$ be satisfied. It is evident that A_n contains no a_ν whose index ν is $>n$. Therefore A_n is an integer function of g of the degree n. The coefficients l_0, l_1, ... l_n are in the present case determined by the equation

$$(x-1)^{-\delta}(x-a)^{n-\alpha+\delta-\beta}=l_0+l_1x+l_2x^2+\ldots.+l_nx^n+\ldots.$$

They are of course independent of g. Consequently the condition $c_n=0$ will become an algebraical equation of the n^{th} degree in respect to the quantity g. Hence the theorem:

*** This manner of denoting integral functions by their table of the indices is adopted from Riemann.**

V. "If, in the differential equation

$$x(x-1)(x-a)\frac{d^2\eta}{dx^2}+[(\alpha+\beta+1)x^2-\{\alpha+\beta-\delta+1+a(\gamma+\delta)\}x+a\gamma]\frac{d\eta}{dx}$$
$$+\alpha\beta(x-g).\eta=0,$$

of which $F[a(\alpha, \beta, \gamma, \delta)g, x]$ is a solution, the third element γ takes the value $\gamma = n+1$ (n being a positive integer), then it is always possible to determine the characteristic parameter g by an algebraical equation of the degree n, so that the point $x=0$ degenerates into a pseudo-singular one. The elements α, β, δ are not subjected to any algebraical condition."

Theorems altogether analogous to the preceding one may be enunciated in reference to the points $x = 1, a, \infty$.

5.

Let us proceed to differential equations of the second order with any number of singular points. Suppose $\xi_1, \xi_2, \dots. \xi_0$ of the series $\xi_1, \xi_2, \dots. \xi_i$ to have degenerated into pseudo-singular points. This hypothesis implies the equation (cf. Theorem III)

$$c_{n_1}=0,\; c_{n_2}=0, \dots. c_{n_v}=0, \tag{1}$$

and subjects, therefore, the $i-2$ characteristic parameters of the equation

$$\psi(x)\frac{d^2\eta}{dx^2}+\chi(x)\frac{d\eta}{dx}+[\mathfrak{l}^{(1)}_{i+1}.\mathfrak{l}^{(2)}_{i+1}.x^{i-2}+\varkappa_1x^{i-3}+\dots.+\varkappa_{i-2}].\eta=0 \tag{2}$$

to v conditional equations.

The original table of the indices of η is

$$\begin{Bmatrix} 0, & 0, & \dots. & 0, & \mathfrak{l}^{(1)}_{i+1} \\ \mathfrak{l}_1, & \mathfrak{l}_2, & \dots. & \mathfrak{l}_i, & \mathfrak{l}^{(2)}_{i+1} \end{Bmatrix}.$$

As soon as

$$\begin{array}{l} \mathfrak{l}_1 = -n_1,\; \mathfrak{l}_2 = -n_2, \dots. \mathfrak{l}_v = -n_v, \\ c_{n_1} = 0,\; c_{n_2} = 0, \dots. c_n = 0, \end{array}$$

only $\xi_{v+1}, \xi_{v+2}, \dots. \xi_i$ remain proper singular points of equation (2), which we write now in the form

$$\phi(x).\psi_1(x)\frac{d^2\eta}{dx^2}+\chi(x).\frac{d\eta}{dx}+\tilde{\omega}(x), \eta=0, \tag{3}$$

making $\phi(x)=(x-\xi_1)(x-\xi_2)\dots.(x-\xi_v)$ and $\psi_1(x)=\dfrac{\psi(x)}{\phi(x)}$.

The last differential equation has not the regular form (A) [No. 1], notwithstanding its general integral η may be expressed by the general integral z of the auxiliary regular differential equation

$$\psi_1(x)\frac{d^2z}{dx^2}+\chi_1(x)\frac{dz}{dx}+\tilde{\omega}_1(x).z=0, \tag{4}$$

where $\chi_1(x)$ and $\tilde{\omega}_1(x)$ denote integer functions of the degrees $i-v-1$, $i-v-2$. The table of the indices of the new function z is supposed to be

$$\left\{\begin{matrix} 0 & , 0 & , \ldots . 0, & \mathfrak{l}_{i+1}^{(1)} \\ \mathfrak{l}_{v+1}, & \mathfrak{l}_{v+2}, & \ldots . \mathfrak{l}_i, & \mathfrak{l}_{i+1}^{(2)}-\sum\limits_{\mathrm{i}=1}^{\mathrm{i}=v} n_i \end{matrix}\right\}$$

The general theory of linear differential equations requires the sum of all the indices in this table to be equal to $i-1$ (cf. Fuchs, Journ. für Mathematik, t. 66, pag. 146). In fact we have

$$\sum_{\mathrm{i}=1}^{\mathrm{i}=i}\mathfrak{l}_\mathrm{i}+\mathfrak{l}_{i+1}^{(1)}+\mathfrak{l}_{i+1}^{(2)}=i-1,$$

but $\sum\limits_{\mathrm{i}=1}^{\mathrm{i}=v}\mathfrak{l}_\mathrm{i}=\sum\limits_{\mathrm{i}=1}^{\mathrm{i}=v}n_\mathrm{i}$, consequently $\sum\limits_{\mathrm{i}=\nu+1}^{\mathrm{i}=i}\mathfrak{l}_\mathrm{i}+\mathfrak{l}_{i+1}^{(1)}+\mathfrak{l}_{i+1}^{(2)}-\sum\limits_{\mathrm{i}=1}^{\mathrm{i}=v}n_\mathrm{i}=i-2$, q. e. d.

The function $\chi_1(x)$ is completely determined by the relations

$$\chi(\xi_\mathrm{i})=-\psi'(\xi_\mathrm{i})(\mathfrak{l}_\mathrm{i}-1)\,|\,\mathrm{i}=v+1,\ v+2,\ \ldots . i.$$

$\tilde{\omega}_1(x)$ must necessarily be of the form

$$\tilde{\omega}_1(x)=\mathfrak{l}_{\mathrm{i}+1}^{(1)}\left(\mathfrak{l}_{i+1}^{(2)}-\sum_{\mathrm{i}=1}^{\mathrm{i}=v}n_\mathrm{i}\right).x^{i-v-2}+\varkappa_1'.x^{i-v-3}+\ldots .+\varkappa_{i-v-2}',$$

$\varkappa_1', \varkappa_2', \ldots . \varkappa_{i-v-2}'$ being the characteristic parameters of the differential equation (4).

Now it is always possible to determine two integer functions $P_0(x)$ and $P_1(x)$ and to express $\varkappa_1', \varkappa_2', \ldots . \varkappa_{i-v-2}'$ by means of $\varkappa_1, \varkappa_2, \ldots . \varkappa_{i-2}$ that the relation

$$\eta=P_0(x).z+P_1(x).\frac{dz}{dx}$$

exists between the functions η and z. The particulars of this method of reduction are fully explained in my paper quoted in the beginning of No. 1 (Acta Mathem. t. 11, pag. 97).

The preceding remarks lead to the subsequent theorem:

VI. "Whenever v of the i zeros $(\xi_1, \xi_2, \ldots . \xi_i)$ of the integral function $\psi(x)$ in the differential equation

$$\psi(x) . \frac{d^2\eta}{dx^2} + \chi(x) . \frac{d\eta}{dx} + \tilde{\omega}(x) . \eta = 0$$

degenerates into pseudo-singular points, the function η will be expressible rationally by a function z satisfying the regular differential equation

$$\psi_1(x) \frac{d^2z}{dx^2} + \chi_1(x) . \frac{dz}{dx} + \tilde{\omega}_1(x) . z = 0$$

where the zeros of $\psi_1(x)$ are the proper singular points of η and its first derivative function $\frac{dz}{dx}$. The $i - v - 2$ characteristic parameters of z are known functions of the $i - 2$ characteristic parameters of η."

Hence we conclude, if $x = 0$ is a pseudo-singular point of the differential equation (2a) [No. 4], the function $F[a(\alpha, \beta, n + 1, \delta) g, x]$ will be expressible by the hypergeometrical series of Gauss.

FRANKFURT A.M., 27 Dec., 1887.

P. S.—M. Poincaré has requested me to correct an error in his memoir, "Sur les groupes des équations linéaires," Acta Math., Vol. IV, p. 217: "Pour qu'un infini des coëfficients ϕ_κ soit un point à apparence singulière, il faut $\frac{(p+2)(p-1)}{2}$ conditions"—this should be ". . . . $2p - 2$ conditions." K. H.